The sensitivity and elasticity analysis of multistate population projections

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Short abstract

Multistate population projections extend the usual projections by age and sex by adding one or more additional dimensions (e.g., educational categories, marital states, health conditions, etc.). The projection results depend on the mortality, fertility and migration rates and numbers as well as the rates of transition between states on which the projections are based. If any parameter is changed, the results of the projection will be different. Our goal is to systematically analyse the links between any projection result (e.g. population sizes, age distributions, dependency ratios) and any age-, sex- or state-specific projection parameter or parameter group on which projections are based. To do so, we present the equations necessary conduct sensitivity and elasticity (proportional sensitivity) analyses of multistate projections, using matrix calculus. We apply our method to a population projection of Germany, which projects the population by age, sex, and citizenship from 2014 to 2050. We identify the parameters which have the largest impact on projected sizes of population subgroups, age groups and ratios. Overall, sensitivity and elasticity analyses provide valuable information on how projection results are driven by mortality, fertility, migration and transition parameters; which parameters deserve particular attention when projection scenarios are prepared; and it may help us to recommend more targeted policy measures to address demographic changes.

Extended abstract

1. Introduction

The results of any population projection depend on the age-, sex- and time-specific fertility, mortality and migration parameters on which they are based. If any of the parameters is modified due to changes in assumptions, modeling techniques, or because new data sources become available, the projection outcomes will be different. Understanding the relation between projection parameters and projection results is important for at least three reasons:

1. It helps to identify which parameters have the strongest impact on projection results, and therefore need to be modeled or estimated most carefully.
2. It provides deeper understanding on how the projection parameters that we feed into projection software collectively shape any projection result that is of interest.
3. It can inform policy makers about which vital rates have the largest impact on future demographic trends and therefore may be suitable targets for policy intervention.

Sensitivity and elasticity analyses are powerful tools to systematically quantify how projection results depend on individual parameters. Sensitivity analysis shows how a projection outcome changes in response to an absolute change in one of the parameters. Elasticity analysis measures the proportional effect of a proportional change in a projection parameter. Here we use matrix calculus to develop systematic sensitivity and elasticity analyses of population projections. Matrix calculus allows us to estimate the effect of small changes ('perturbations') in any projection parameter on any projection outcome that is of interest.

In a recent paper we already presented the sensitivity and elasticity of projection results to parameters and initial conditions for population projections classified by age and sex (Caswell and Sánchez Gassen 2015). For many research purposes and policy questions, however, populations are disaggregated along further dimensions. Multistate population projections provide such detailed projection results. Recent publications have projected populations by education (Lutz, Butz and Samir KC 2014), religion (Stonawski et al 2015), ethnicity (Rees et al. 2012) or citizenship (Sánchez Gassen 2015), in addition to age and sex. We refer to these additional dimensions of population disaggregation as 'states'. Multistate population projections are based on an even larger number of parameters than age- and sex-specific projections. For instance, for a projection covering 100 age groups and 30 projection years (as well as two sexes), 18.000 fertility, mortality and migration parameters have to be defined. If the population is further subdivided into two or more states, the number of these parameters at least doubles. In addition, parameters have to be defined to account for possible transitions between states, e.g. when individuals change their citizenship, religion or achieve a higher educational degree. Overall, the results of multistate population projection are sensitive to an even larger number of parameters than age- and sex-specific projections. Understanding the links between changes in parameters and outcomes is therefore of particular relevance for multistate projections.

In this paper, we will develop the full set of equations necessary to conduct systematic sensitivity and elasticity analyses for multistate population projections, using the vec-permutation matrix approach. This approach permits the systematic construction and analysis of multistate models (Hunter and Caswell 2005, Caswell 2012). Its name refers to the central role of vec-permutation matrices in rearranging population vectors by age and state during the projection and sensitivity calculations. We will illustrate the use and the insights gained from sensitivity and elasticity analyses of multistate population projections with an age-, sex- and citizenship-specific population projection of Germany.

2. *Multistate Population Projections as matrix operators*

We start by formulating projections as matrix operators. As is customary, male and female populations are projected separately. Projections that disaggregate the population by age and sex can be expressed as follows.

$$n_f(t+1) = [U_f(t) + \emptyset F(t)]\, n_f(t) + b_f(t) \qquad n_f(0) = n_0 \quad (1)$$

$$n_m(t+1) = U_m(t)\, n_m(t) + (1-\emptyset)F(t)\, n_f(t) + b_m(t), \qquad n_m(0) = n_0 \quad (2)$$

where $n_f(t)$ and $n_m(t)$ are the male and female population vectors in projection year t. Both are column vectors whose elements contain the number of persons in each age class. In order to project the female population vector one time-step to t+1, we multiply with survival matrix $U_f(t)$, which has age-specific survival rates on the sub-diagonal and zeros everywhere else. We also multiply the population vector with fertility matrix $F(t)$, which has age-specific fertility rates on the first row and zeros everywhere else. A proportion $\emptyset$ of the births are female. Finally, age-specific migration numbers are added or subtracted via column vector $b_f(t)$. The male population vector is projected in a similar manner by multiplying with survival matrix $U_m(t)$ and adding or subtracting migration vector $b_m(t)$. Offspring comes from the fertility of the female population. A proportion $(1-\emptyset)$ of the offspring is male. Projections start with initial population vectors $n_f(0)$ and $n_m(0)$ and are reiterated for each projection year.

Multistate population projections can be calculated in a similar manner. They are more complex, however, since they have to account for different states and transitions between them. We define:

$$\tilde{n}_f(t+1) = [\tilde{U}_f(t) + \emptyset \tilde{F}(t)]\, \tilde{n}_f(t) + \tilde{b}_f(t) \qquad \tilde{n}_f(0) = \tilde{n}_0 \quad (3)$$

$$\tilde{n}_m(t+1) = \tilde{U}_m(t)\, \tilde{n}_m(t) + (1-\emptyset)\tilde{F}(t)\, \tilde{n}_f(t) + \tilde{b}_m(t). \qquad \tilde{n}_m(0) = \tilde{n}_0 \quad (4)$$

Here, population vectors $\tilde{n}_f(t)$ and $\tilde{n}_f(t)$ contain the number of persons in each age group and state. Individuals belong to states 1, …, s and age groups 1, …, w. The population vectors then take the form

$$\tilde{n}(t) = \begin{pmatrix} n_{11} \\ \vdots \\ \underline{n_{w1}} \\ \vdots \\ n_{1s} \\ \vdots \\ n_{ws} \end{pmatrix} (t) \quad (5)$$

The vector $\tilde{n}(t)$ is of dimension $sw \ x \ 1$ and shows the number of persons per age, arranged within states. Migration vectors $\tilde{b}_f(t)$ and $\tilde{b}_m(t)$ take a similar form. They contain the number of net migrants in each age group, arranged within states, and are also of dimension $sw \ x \ 1$:

$$\tilde{b}(t) = \begin{pmatrix} b_{11} \\ \vdots \\ b_{w1} \\ \hline \vdots \\ b_{1s} \\ \vdots \\ b_{ws} \end{pmatrix}(t) \qquad (6)$$

Matrices $\tilde{U}_m(t)$ and $\tilde{U}_f(t)$ in equations (3) and (4) contain survival rates and transition rates between states. They are defined as

$$\tilde{U}(t) = K^T VKU. \qquad (7)$$

When multiplying matrices $\tilde{U}_m(t)$ and $\tilde{U}_f(t)$ with population vectors $\tilde{n}_f(t)$ and $\tilde{n}_m(t)$, we go from right to left: First, matrix U of dimension $sw \ x \ sw$ contains survival rates on the subdiagonal and zeros elsewhere, and moves all individuals to the next highest age class. Then, the vec-permutation matrix K of dimension $sw \ x \ sw$ rearranges the result vector so that states are arranged within age groups. The vec-permutation matrix K contains zeros and ones; a formula is given in Hunter and Caswell (2005).

After rearranging, we multiply by block diagonal matrix V of dimensions $sw \ x \ sw$. This matrix contains transition rates and moves individuals between states without changing their age. Finally, the transposed vec-permutation matrix K^T reorganizes the resulting vector again, so that ages are again arranged within states.

The matrix $\tilde{F}(t)$ takes a similar structure:

$$\tilde{F}(t) = K^T HKF. \qquad (8)$$

When $\tilde{F}(t)$ is multiplied with the female population vector $\tilde{n}_f(t)$, we again proceed from right to left and first multiply with fertility matrix F of dimension $sw \; x \; sw$. This matrix contains age-specific fertility rates for each state. Then, we rearrange the resulting vector with vec-permutation matrix K, so that states are arranged within age groups. We multiply with matrix H next, which contains transition rates that allocate newborns to the different states. Finally, K^T rearranges the resulting vector back to the original form, with age groups arranged within states.

Projection equations (3) and (4) are repeated for each projection step, starting with initial population vectors $\tilde{n}_m(0)$ and $\tilde{n}_f(0)$. The resulting population vectors contain information on the number of men and women in each projection year by age group and state. This information can be condensed to show any outcome that might be of interest such as the total size of the population; the size of different states; the number of persons of a given age span in each state; population ratios etc.

3. Perturbation analysis of projections

The outcome of multistate population projections depends on the age- and sex-specific fertility, mortality and migration parameters that have been defined for each projection year, as well as on parameters that define transitions between states and allocate newborns to states. To quantify how a change in any of these input parameters influences projection results, we use differential analysis. If projection result y is a function of parameter x, the sensitivity and elasticity (proportional sensitivity) of y with respect to changes in x are

Sensitivity: $$\frac{dy}{dx} \quad (9)$$

and Elasticity: $$\frac{\epsilon y}{\epsilon x} = \frac{x}{y}\frac{dy}{dx} \quad (10)$$

The projection results of interest here are the projected population vectors $\tilde{n}_f(t)$ and $\tilde{n}_m(t)$ or any variable that can be derived from them. To quantify the sensitivity and elasticity of these vectors in any projection year t with respect to a change in parameter vector θ in any projection year s, we calculate for the female population vector

$$\text{Sensitivity:} \quad \frac{d\tilde{n}_f(t)}{d\theta^T(s)} \tag{11}$$

$$\text{and} \quad \text{Elasticity:} \quad \frac{\epsilon\tilde{n}_f(t)}{\epsilon\theta^T(s)} = diag\left(\tilde{n}_f(t)\right)^{-1} \frac{d\tilde{n}_f(t)}{d\theta^T(s)}\, diag(\theta(s)) \tag{12}$$

and similar for the male population vector. Parameter vector $\theta^T(s)$ is the transpose of parameter vector $\theta(s)$. It can contain age-, sex- and state-specific mortality rates μ, fertility rates f, migration numbers b, transition rates between states v or state allocation probabilities of newborns h, depending which parameter group is of interest.

The sensitivity and elasticity results in (11) and (12) are calculated iteratively, over the entire projection period, in parallel with the projection itself. Caswell and Sánchez Gassen (2015) derived results for the case of age-sex classified projections. Our analyses here generalize those results to multistate projections using the vec-permutation model. The results are applicable to all multistate models which project a population by age, sex and any number of additional states. They provide the sensitivity and elasticity of projection outcomes with respect to any age-, sex- or state-specific parameter or parameter group on which they are based.

4. *First results - Population projection by age, sex and citizenship*

To illustrate the perturbation analysis of multistate population projections, we use a recent projection by Sánchez Gassen (2015). It projects the population of Germany by single-year age groups, sex and citizenship (German/non-German) and covers a projection period from 2010 to 2030. Here, we updated the projection to cover an extended time period between 2014 and 2050. The projection contains 17 projection scenarios, each based on a different combination fertility, mortality and migration parameters for Germans and foreign residents, as well as on different scenarios for future trends in citizenship changes (naturalization). In the most basic 'reference' scenarios all vital rates were kept constant during the projection period at the level observed in 2014. According to this scenario, the number of German citizens living in Germany will decline during the coming decades, from 73.8 million (2014) to 62.4 million persons (2050). The number of foreign residents, by contrast, is expected to increase from 7 million persons in 2014 to 10.5 million persons in 2050. How sensitive are these projection results change to changes in the projection parameters? Figure 1 shows the elasticity of the projected size of the German (upper panels) and the foreign population (lower panels) in Germany in 2050 to perturbations in parameters, applied in every projection year. In other words, we measure how the size of these population groups changes if we increase all male and female age-specific fertility, mortality, migration and naturalization parameters by the same small proportion between 2014 and 2050.

The x-axes of the figures show the age at which we perturb the demographic rates and numbers; the y-axis measures the size of the effect. Positive values on the y-axis indicate that the size of the population groups in 2050 increases if we change a parameter; negative values indicate that a perturbation in parameters reduces the final population size. The upper left panel shows the elasticity of the German population with respect to perturbations in the German vital rates; the upper right panel shows how the size of the German population responds to perturbations in foreign residents' vital rates. The German population size is sensitive to changes in demographic behaviours of the foreign population, because the offspring of foreign couples may acquire the German citizenship at birth; and because a proportion of foreign residents become German citizens each year. The lower right panel shows the elasticity of the foreign population to changes in foreign residents' vital rates; the lower left panel (elasticity of the foreign population to changes in German rates) remains empty. Changes in vital rates of the German population have no effect on the foreign population, since the projection model assumes no transitions from the German to the foreign population. In other words, it is assumed that Germans do not give up their citizenship in favor of acquiring a foreign citizenship while remaining resident in

Germany. Any changes in the demographic behavior of Germans therefore have no effect on the future development of the foreign population.

Two first insights can be gained from the figures: First, the foreign population is more sensitive to perturbations in the vital rates than the German population. This is particularly the case for perturbations in immigration numbers and emigration rates. Here, Figure 1 shows much stronger effects for the foreign population than for the German population. Second, the German and the foreign population respond to different degrees to different kinds of perturbations: The foreign population is most sensitive to perturbations in immigration numbers and emigration rates, particularly around the ages 20 to 30. A one percent increase in immigration numbers around age 25 in every single projection year, for instance, increases the size of the foreign population in 2050 by around 0.35 percent. No other parameter change has a similarly strong impact on the projected population size of this group. For the German population, perturbations in fertility rates, particularly around age 30, and of mortality rates, especially around age 80, have the strongest effect. Perturbations in fertility increase the final population size, while perturbations in mortality lead to the faster attrition of population members and thereby reduce the population size. Changes in German and foreign immigration and emigration parameters also influence the size of the German population, albeit to a weaker degree.

In order to estimate the sensitivity of different population subgroups to perturbations in projection parameters, the whole set of parameters has to be taken into account. If individuals can change their state membership during the projection period, then each population group will not only be sensitive to perturbations of their own rates, but also to perturbations in the vital rates of other subgroups. The extent of these cross-sensitivities depends on the strength of transition rates between states, but may also depend on the projection outcome that is of interest. As a next step, we will analyze these links in a systematic manner, by applying sensitivity and elasticity analyses to a broader range of projection scenarios. We will also analyze the sensitivity and elasticity of a broader range projection results that can be calculated on the basis of population vectors $\tilde{n}_f(t)$ and $\tilde{n}_m(t)$ – such as the size of age groups, population ratios, etc.

Overall, this study will develop the full set of equations necessary to conduct a sensitivity and elasticity analysis of multistate population projections. These analyses will broaden our understanding of how projection parameters collectively produce the projection results we observe. It will also allow us to identify the parameters that have the largest impact on projection results, and therefore deserve particular attention from statisticians who prepare projections; and from policy makers who consult projections to inform policies and programs.

5. *Literature*

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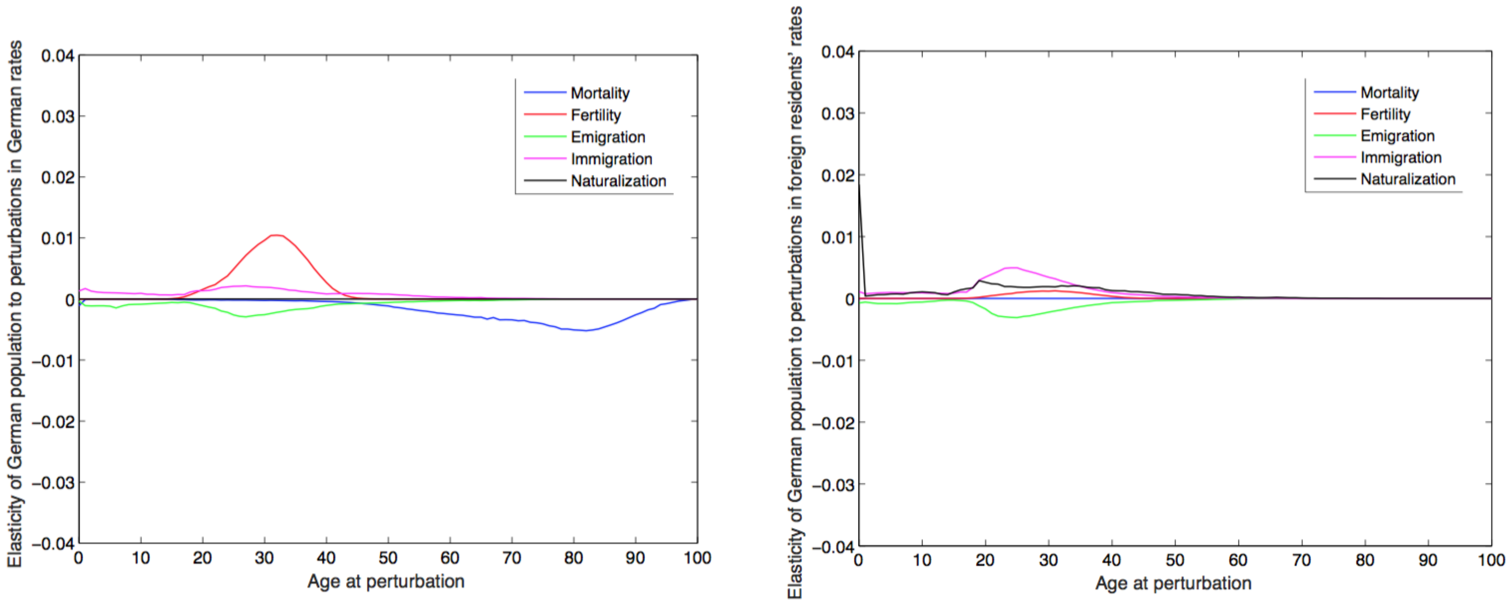

Figure 1(a): The elasticity of the German population to perturbations in German vital rates (left side) and foreign residents' vital rates (right side)

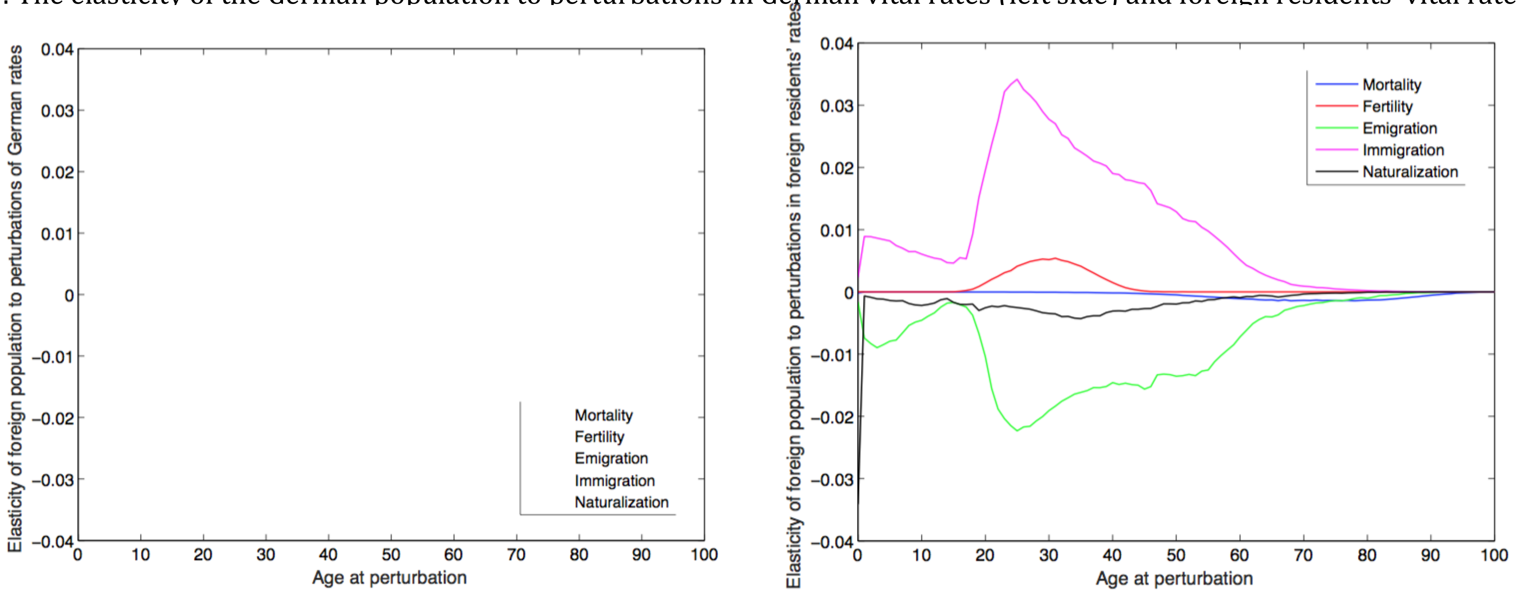

Figure 1(b): The elasticity of the foreign population to perturbations in German vital rates (left side) and foreign residents' vital rates (right side)